Problem 4: Zeckendorf cover - Solution

Let us note that in the problem statement $n \leq 1000$, so calculating all fibonacci covers which is of order $n!$ is unfeasible. We need to something cleverer.
The first observation to make is that the exact values of the Fibonacci numbers do not matter for this problem. We simply need to find the number of ways to paritition the set $S_n = \{1, ..., n\}$, where each class of our partition contains no consecutive integer.

Let us denote all such partitions of S_n by Z_n. Then, we want to know $|Z_n|$. Now, suppose we have Z_n, we can than find Z_{n+1} by taking $n+1$ and adding it in all possible ways to partitions of S_n.
Now, suppose P is a partition of S_n, then we have two options to create a partition of S_{n+1}: firstly we can add $\{n+1\}$ to P. The other options is that we add $n+1$ to a set in P. However, note we cannot add $n+1$ to the set in P that contains n, as then we would have a partition with consecutive integers. So, in total there $1 + (|P| - 1) = |P|$ ways of constructing a partition of S_{n+1} from a partition of S_n. Also note that all partitions created in this way from partitions of S_n are different and that any partitions of S_{n+1} can be constructed in such a way. So, we see that

$$|Z_{n+1}| = \sum_{P \in Z_n} |P|.$$

Now, this sum counts the total number of sets in all the partitions of S_n combined.

Note that to find all partitions P in Z_n is order $n!$. Instead, we will ask the following question, how many partitions are there with k elements? Let us denote this number by $N_n[k]$.

$$Z_n = \sum_{k=1}^{n-1} k N_{n-1}[k].$$

Now, observe that the first option of adding the element $\{n+1\}$ to P increases the size of P by 1, so if $|P| = k-1$, it contributes exactly once to $N_n[k]$. On the other hand, observe that the second option does not increase the size of P, but there are $|P| - 1$ ways of add $n+1$, so we get that if $|P| = k$, it contributes $k-1$ times to $N_n[k]$. Thus, we get that

$$N_n[k] = (k-1)N_{n-1}[k] + N_{n-1}[k-1].$$

Now, we can calculate the table $N_n[k]$ using the initial condition $N_0[0] = 1$ and $N_0[k] = 0$ otherwise. When writing your implementation, take care to do modulo with $10^9 + 7$ often to prevent numbers in your calculation from becoming too large. Furthermore pay attention to the boundary cases such as $n = 1$.